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Orderings of one-dimensional Ising systems with an arbitrary interaction of finite range II

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Abstract. A study is made of the ground state orderings of the one-dimensional Ising system, under the assumption that the interaction is of finite range r and has inversion symmetry. It is shown that the ground state energy is effected by the regular chain either of an irreducible block without or with one symmetry point of r or $(r+1)$ sites, or of a symmetric irreducible block with two symmetry points. For the case of the one-dimensional Ising magnet under a uniform external field, it is effected by two, four, seven and twenty regular orderings, when the range r of the interaction is one, two, three and four, respectively.

Morita and Horiguchi (1972) gave an elementary argument to prove that the ground state energy of the one-dimensional Ising magnet of spin $\frac{1}{2}$ with an interaction up to next-nearest neighbours, is effected by four orderings under a uniform external field. In a recent paper (Morita 1974), the argument was generalized to the system with an interaction of longer but finite range r, with the aid of the concepts of *reducible* and *irreducible blocks*. It was shown that an irreducible block consists of at most s_M^r lattice sites and the total number of different irreducible blocks is finite. Here s_M is the total number of possible configurations for each lattice site and it is assumed to be finite. It was also shown that the total energy E and the total number of lattice sites L of an arbitrary arrangement of the chain is expressed as follows:

$$E = \sum_{\mathrm{B}} n_{\mathrm{B}}\epsilon_{\mathrm{B}}, \tag{1}$$

$$L = \sum_{\mathrm{B}} n_{\mathrm{B}}\nu_{\mathrm{B}}, \tag{2}$$

where the summation is taken over all the irreducible blocks B, and n_{B} are zero or positive integers; ϵ_{B} and ν_{B} being the energy and the number of lattice sites per block in the regular chain of the irreducible block B. It was concluded from (1) and (2) that the ground state energy of the system is effected by the regular chain of the irreducible block B for which $\epsilon_{\mathrm{B}}/\nu_{\mathrm{B}}$ is a minimum. The inversion symmetry of the system was not exploited.

Bundaru *et al* (1973) generalized the argument of Morita and Horiguchi (1972) to the Ising magnet with interaction of range three. They showed that some of the irreducible blocks can be excluded from the set of the blocks which give the ground state, because of the inversion symmetry of the lattice. It is the purpose of the present paper to supplement the result of the preceding paper (Morita 1974), by taking account of the inversion symmetry of the system.

We shall use the word *symmetry point* in the following sense. In a chain, it is a point of the inversion symmetry for r or $(r+1)$ sites around that point. In a reducible or an irreducible block, it is a point which becomes a symmetry point in the regular chain of that block. When a reducible or an irreducible block involves two symmetry points and when both of these points are the points of inversion symmetry of the regular chain of the block, we call that block a *symmetric block*. For example, the block $(+++-+--)$ composed of $+$ and $-$ involves four symmetry points if $r=3$; they are at the positions of the second $+$, first $-$ and fourth $+$, and at the middle point between the last two $-$. If $r=4$, the same block involves two symmetry points; they are at the position of the second $+$ and at the middle point between the last two $-$. This block is not a symmetric one.

There are s_M^r different arrangements for r sites. Among them, $s_M^{[(r+1)/2]}$ are symmetric ones, where $[(r+1)/2]$ is the integral part of the ratio $(r+1)/2$. An irreducible block of l sites, involves l sets of successive r sites, which are all different from each other. Among these l sets, at most $s_M^r - s_M^{[(r+1)/2]}$ can be asymmetric ones, and hence we have at least $l - s_M^r + s_M^{[(r+1)/2]}$ symmetric successive r sites, if $l > s_M^r - s_M^{[(r+1)/2]}$. As the central point of each of the symmetric successive r sites is a symmetry point, we have the following lemma.

Lemma 1. In an irreducible block of l sites, there occur at least $l - s_M^r + s_M^{[(r+1)/2]}$ symmetry points, if l is larger than $s_M^r - s_M^{[(r+1)/2]}$.

When p is a symmetry point of a chain, we shall denote the r or $(r+1)$ lattice sites around the p by $S(\mathrm{p})$. Then we have the following lemma.

Lemma 2. If p is a symmetry point in a chain, the total energy E of that chain is expressed as

$$E = E[J_{\mathrm{L}} \cup S(\mathrm{p})] + E[S(\mathrm{p}) \cup J_{\mathrm{R}}] - E[S(\mathrm{p})].$$

Here $E[\mathrm{C}]$ is the total energy of the part C of the chain, $\mathrm{C}_1 \cup \mathrm{C}_2$ denotes the part which consists of all the lattice sites involved either in the part C_1 or in C_2, and J_{L} and J_{R} are the parts of the chain to the left and to the right, respectively, of p. We have a similar relation for the total number of lattice sites.

We shall now consider the regular chain of an irreducible block B which involves n symmetry points, where $n \geqslant 2$; if $n=2$, we consider only the case when the irreducible block is not symmetric. The regular chain of the block is schematically expressed as figure 1(*a*), where $\mathrm{p}_1, \mathrm{p}_2, \ldots,$ and p_n, respectively, denote the replicas of the n symmetry points, $\mathrm{I}_1, \mathrm{I}_2, \ldots,$ and I_{n-1} are the parts connecting the points p_1 and p_2, p_2 and $\mathrm{p}_3, \ldots,$ and p_{n-1} and p_n, successively, and the part I_n connects the points p_n and p_1. By a repeated use of lemma 2, we can express the energy ϵ_{B} and the number of sites ν_{B} per block in this chain as follows:

$$\epsilon_{\mathrm{B}} = \epsilon'_1 + \epsilon'_2 + \ldots + \epsilon'_n, \tag{3}$$

$$\nu_{\mathrm{B}} = \nu'_1 + \nu'_2 + \ldots + \nu'_n, \tag{4}$$

where ϵ'_i is given by

$$\epsilon'_i = E[S(\mathrm{p}_i) \cup \mathrm{I}_i \cup S(\mathrm{p}_{i+1})] - \tfrac{1}{2}E[S(\mathrm{p}_i)] - \tfrac{1}{2}E[S(\mathrm{p}_{i+1})],$$

and ν'_i is given by a similar expression for the number of lattice sites; ν'_i is either an integer or a half-odd integer.

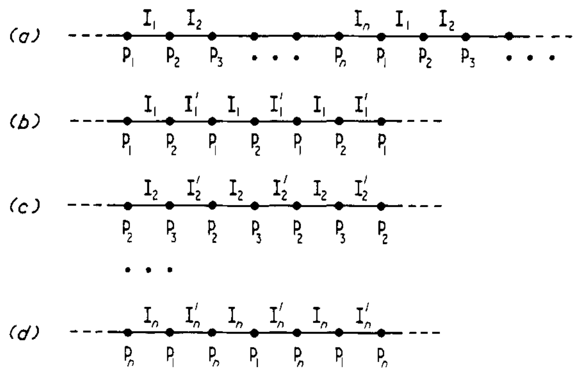

Figure 1. (*a*) The regular chain of an irreducible block B with n symmetry points ($n \geqslant 2$). (*b*), (*c*), ..., (*d*) n regular chains of symmetric blocks, which are made of the parts involved in the original block B.

Here we shall construct n chains, as shown in figures 1(*b–d*). They are constructed of the parts appearing in the chain shown in figure 1(*a*). The part $I_k'(k = 1, 2, \ldots, n)$ is obtained from the part I_k by the process of inversion around p_k or p_{k+1}, where p_{n+1} represents p_1. The kth chain is the regular chain of the symmetric block which consists of one of each of I_k, I_k', p_k and p_{k+1}. It should be noted that p_k and p_{k+1} are the symmetry points of the block. By applying lemma 2 to these chains, we note that two times ϵ_k' and two times ν_k' are equal to the energy and the number of lattice sites per block in the kth chain, by virtue of the inversion symmetry of the interaction. If a reducible block occurs among the n symmetric blocks, it is further decomposed into a sum of irreducible blocks, which can involve either zero or one symmetry point. As a result, ϵ_B and ν_B are expressed as follows:

$$\epsilon_B = \tfrac{1}{2}\sum_{D}{}' n_D^{(B)}\epsilon_D, \tag{5}$$

$$\nu_B = \tfrac{1}{2}\sum_{D}{}' n_D^{(B)}\nu_D, \tag{6}$$

where the primes denote that the summations with respect to D are taken over the irreducible blocks without or with one symmetry point and the symmetric irreducible blocks with two symmetry points, and $n_D^{(B)}$ are zero or positive integers.

In equations (1) and (2), ϵ_B and ν_B for the irreducible blocks with more than two symmetry points and for the asymmetric irreducible blocks with two symmetry points can be expressed by means of (5) and (6), respectively. Then the total energy and the total number of the lattice sites of a chain in an arbitrary arrangement are expressed as follows:

$$E = \tfrac{1}{2}\sum_{B}{}' n_B'\epsilon_B, \tag{7}$$

$$L = \tfrac{1}{2}\sum_{B}{}' n_B'\nu_B, \tag{8}$$

where n_B' are zero or positive integers. The minimum of E is achieved by putting $n_B' = 2L/\nu_B$ for the irreducible block B without or with one symmetry point or for the symmetric irreducible block B with two symmetry points, for which ϵ_B/ν_B is a minimum.

The minimum energy is effected by the regular chain of that irreducible block B. Thus we get the following theorem.

Theorem. The ground state energy is effected by the regular chain either of an irreducible block without or with one symmetry point, or of a symmetric irreducible block with two symmetry points.

We now need only the irreducible blocks with two or less symmetry points. Lemma 1 shows that they consist of $s_M^r - s_M^{[(r+1)/2]} + 2$ or less lattice sites. If $s_M = 2$, this number is 4 which is equal to s_M^r, for $r = 2$, and it is 6 less than $s_M^r = 8$ for $r = 3$, and 14 less than $s_M^r = 16$ for $r = 4$.

All the irreducible blocks which we have to consider in searching for the ground state are as follows for the case $s_M = 2$ and $1 \leqslant r \leqslant 4$. We have [1], [12] for $r = 1$, and further [112], [1122] for $r = 2$, and further [1112], [11122], [111222] for $r = 3$, and further [11112], [11212], [111212], [111122], [1111212], [1111222], [11212212], [11112222], [111212212], [1111212212], [1112122212], [11112122212], [111121222212] for $r = 4$. We list here those in which the total number of 1 is equal to or more than that of 2. Note that they are all symmetric blocks with two symmetry points. From this list, we see that there are two, four, seven and twenty spin orderings for the one-dimensional Ising magnets with an interaction of $r = 1, 2, 3$ and 4, respectively. The seven for $r = 3$ were found by Bundaru *et al* (1973) and Katsura and Narita (1973); the latter authors gave the figures to show when each of the orderings occurs.

If $s_M = 3$, we have all those given for the case of $s_M = 2$. In addition to them, we have [123] for $r = 1$, and further [1213], [1123], [11232], [121323], [112332], [123132], [1123132] for $r = 2$. In this list, those which are obtained from the listed ones by a permutation of the numbers 1, 2 and 3 are not given. We note that [123] involves no symmetry point and [1123] only one symmetry point if $r = 2$.

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